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On artificial geometry errors in the adaptive analysis of eigenvalues and eigenmodes of curved shell structures using the finite element method

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The standard procedure to compute design loads for shell structures as also proposed in design rules is based on the computation of the limit load taking into account the modification of the so-called stability loads due to geometrical imperfections. The imperfections are mostly chosen affine to the buckling patterns, which are solutions of the eigenvalue-problem for the geometrically perfect structure. Thus, the eigenvalue-problems for stability points have to be solved very accurately. In the present contribution an adaptive h-refinement procedure is taken for the solution using low order shell elements.

1 Adaptive computation of stability points

After discretization with finite elements and linearization within a Newton type solution the following system of equations must be solved to compute states of equilibrium

$$\boldsymbol{K}_T(\boldsymbol{u}_i)\Delta\boldsymbol{u}_{i+1} = \lambda\boldsymbol{p} - \boldsymbol{r}(\boldsymbol{u}_i) \quad \text{with} \quad \boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta\boldsymbol{u}_{i+1}. \tag{1}$$

Herein $\boldsymbol{K}_T$ denotes the tangent stiffness matrix, $\boldsymbol{p}$ the load vector, λ the load multiplier and $\boldsymbol{r}(\boldsymbol{u}_i)$ the internal forces. $\Delta\boldsymbol{u}_{i+1}$ is the vector of the displacement increments for the next iteration step. The tangent stiffness matrix can be used in stability investigations of converged states of equilibrium.
As it is well-known for conservative systems as considered here, a state of equilibrium is stable, if $\boldsymbol{K}_T$ is positive definite, i.e. all eigenvalues μ from the standard eigenvalue problem

$$(\boldsymbol{K}_T - \mu_k\boldsymbol{I})\,\boldsymbol{\Psi}_k = \boldsymbol{0} \tag{2}$$

are larger than zero. Hence, the transition from stable to unstable equilibrium states is characterized by zero eigenvalues. The lowest eigenvector $\boldsymbol{\Psi_1}$ computed at the singular point gives then the buckling mode.
An adaptive h-refinement procedure might improve the convergence behavior of a FE solution. The energy norm of the error is defined as

$$||\boldsymbol{e}_h||^2 = \int_{\Omega_0} (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h) : (\boldsymbol{\epsilon}^* - \boldsymbol{\epsilon}_h) d\Omega. \tag{3}$$

with discrete values from the FE solution, which are denoted with "h". The values denoted with "*" are called recovered values and are computed by a least-square fit on element patches using the so-called superconvergent patch recovery procedure, see Zienkiewicz and Zhu [2].
Combination of the stability criterion with the error estimator leads to the following procedure in an adaptive computation of singular points and buckling modes: starting with an initial mesh, e.g. 20x20 elements, singular points are computed using a nonlinear computation monitoring the lowest eigenvalue of (2) in combination with the bisection procedure. Then, the error distribution is computed using (3) with the buckling mode $\boldsymbol{\Psi_1}$ instead of the displacement vector, similar to the approach of Stein et al. [1]. But in the present work the system matrices of the unloaded case are used, i.e. the modifications due to nonlinearities are neglected, because the investigated problem is rather linear in the pre-buckling state. Finally a new mesh is achieved refining the elements with an error greater than 1/4 of the maximum error in the model. Then, the next refinement step is performed for the refined mesh starting with the computation of a new singular point.

2 Cylinder under axial compression

Mesh convergence studies are performed for a quarter of a perfect cylinder under axial compression using uniform and adaptive mesh refinement. The cylinder consists of steel with $E = 2 \cdot 10^5\ N/mm^2$ and $\nu = 0.3$. The geometry data are: radius $R = 625\ mm$, height $H = 966\ mm$ and thickness $t = 0.56\ mm$. As boundary conditions the upper and lower edges are hinged allowing displacements in axial direction. At both edges a defined load is applied in axial direction. In this case the well-known analytical solution can be written as

$$\alpha = \frac{F_{cr}}{F_{cr,cl}} = 0.843 \quad \text{with} \quad F_{cr,cl} = \frac{2\pi E t^2}{\sqrt{3(1-\nu^2)}}. \tag{4}$$

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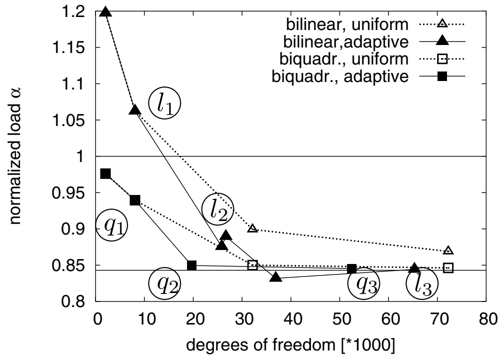

Fig. 1 Static buckling loads of a cylinder with perfect geometry modeled with bilinear and biquadratic elements; normalized to the classical critical load; uniform and adaptive refinement.

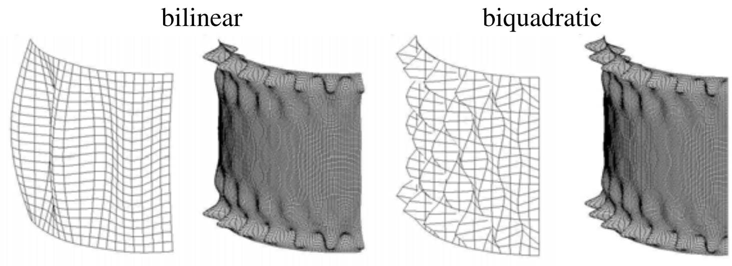

Fig. 2 First eigenvectors $\boldsymbol{\Psi}_1$ at singular points for coarse (N_{dof} = 2040) and fine (N_{dof} = 72240) meshes; uniform refinement with bilinear and biquadratic elements.

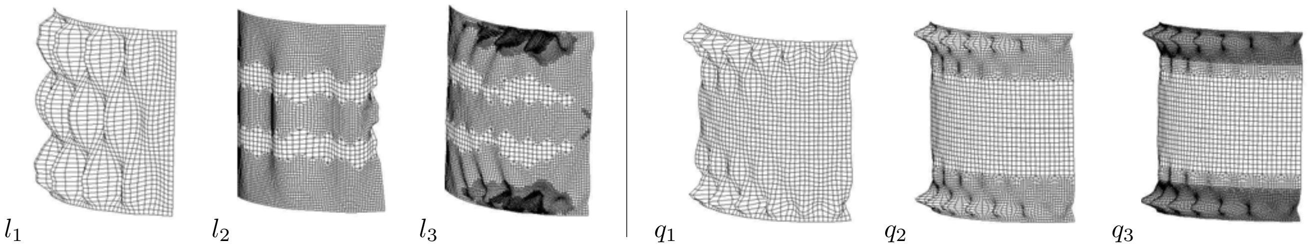

Fig. 3 First eigenvectors $\boldsymbol{\Psi}_1$ at singular points for refinement steps l_1, l_2, l_3 and q_1, q_2, q_3 indicated in Fig.1; adaptive refinement.

with the normalized critical load α for the given boundary conditions, the critical load F_{cr} and the classical critical load $F_{cr,cl}$. Two different shell elements are used in the FE model: a bilinear shell element with ANS for the transversal shear strains and a biquadratic shell element MITC9, both proposed by Bathe et al. As expected the results from the analysis with uniform refinement – dashed lines in in Fig. 1 – show a much better convergence for the biquadratic elements than for bilinear elements. Though different eigenvectors are achieved for coarse meshes, both models converge to the same final buckling mode, see Fig. 2.

Within the adaptive analysis the procedure described before is applied to bilinear shell elements. In the case of biquadratic elements the mesh has been refined at the edges of the cylinder twice to check the influence of transition elements. The results of the adaptive analysis show, that the convergence behavior concerning the buckling load could be improved with non-uniform refinement, see Fig. 1. However, for the model with the bilinear elements the computed buckling modes do not converge to the target mode of the uniformly refined mesh, see Fig. 3 ($l_1 - l_3$). This is due the bad geometry approximation by the transition elements, which appear in adaptive meshes. They are always distorted and mostly considerably warped for curved shells. In particular they often cause artificial geometrical imperfections leading to buckling modes, which are partially totally different from the correct modes. Modeling the cylinder with biquadratic elements the transition elements lead to almost negligible artificial imperfections with no visible influence on the computation of eigenmodes at singular points, see Fig. 3 ($q_1 - q_3$).

3 Conclusions

The convergence behavior of the solution for bilinear shell element could be considerably improved using the proposed adaptive procedure for computation of buckling load. In the case of linear approximation of the geometry warped transition elements introduce often artificial geometrical imperfections, leading to wrong modes. It is an indication that buckling analyses using an approximation by bilinear elements should be performed with great care preferably with uniformly refined meshes. For non-uniform meshes and curved geometry – not unexpected – biquadratic shell elements prove to be the clearly better choice.

References

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